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RANDOM WALK, SCATTERING AND
INVARIANT IMBEDDING—I:
ONE-DIMENSIONAL DISCRETE CASE

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SUMMARY

In this paper we introduce a new method of treating problems involving random walk processes, based upon the principle of invariant imbedding which we have introduced and applied in previous papers. Since scattering processes can often be formulated in terms of random walk, we have a new method of treating scattering processes.

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1. INTRODUCTION

It is well known that a large variety of important physical processes can be formulated in terms of random walk, cf. Chandrasekhar⁵. In particular, multiple scattering may be discussed in these terms, cf. Wigner⁹.

In a sequence of papers, 1, 2, 3, 4 we have introduced and presented applications of a general principle of invariance which we have called the "principle of invariant imbedding". This is a simultaneous distillation and extension of the various invariance principles used by Chandrasekhar in his book⁶, where references to earlier uses by Ambarzumian are given. In this paper we wish to present some further applications of this fundamental approach, turning our attention now to the fields of random walk processes and scattering.

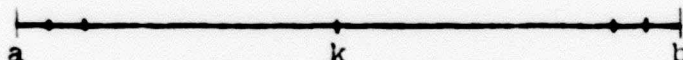
In order to illustrate the underlying ideas as clearly as possible, unhampered by purely technical detail, we shall consider two one-dimensional discrete processes. In subsequent papers, we shall apply the same methodology to corresponding problems for multi-dimensional regions and to continuous versions. Since continuous versions of random walk processes lead to the heat equation and the potential equation, we are led via this route to a new analytic treatment of these classical equations.

Finally, let us note that as in Chandrasekhar's work on

radiative transfer⁶, these principles of invariance give rise to new methods of computational solution.

2. INHOMOGENEOUS RANDOM WALK WITH TWO ABSORBING BARRIERS

Consider a stochastic process in which a moving particle can occupy any one of the lattice points on the line segment $[a, b]$,



$k = a + 1, a + 2, \dots, b - 1$. If the particle is at k at any time t , with probability $p(k)$ it will be at $k - 1$ at time $t + 1$, and with probability $q(k)$ it will be at $k + 1$ at time $t + 1$.

We wish to determine the probability that a particle starting at a point x at $t = 0$ will land at the point a before landing at b . This is an inhomogeneous version of the "gambler's ruin" problem.

The classical treatment of problems of this nature, cf. Chandrasekhar⁵, Uspensky⁷, regards a and b as fixed quantities and x as a variable quantity. The problems are then resolved by means of recurrence relations involving a function $u(k)$. Another approach to questions of this type, also keeping a and b fixed, is due to Wald⁸, using his "fundamental lemma." Problems of this nature arise frequently in the theory of sequential analysis.

Here we wish to present an alternative treatment which

regards x and b as fixed and a as variable. It will be based upon the invariance principles mentioned in the foregoing section.

3. INVARIANT IMBEDDING

It is clear that the probability we wish to determine depends upon a , b , and x . Keeping b fixed, we introduce the function $f(a;x)$, defined for $a < x < b$, as the required probability that a particle starting at x lands at a before landing at b .

In order to obtain the functional dependence upon a , we make the observation that the only way in which the particle can arrive at a before arriving at b is for it to land at $a + 1$ before landing at b , and then, starting from $a + 1$, to proceed to a before ever landing at b .

The mathematical translation of this logical decomposition is the functional equation

$$f(a;x) = f(a+1;x)f(a;a+1). \quad (3.1)$$

4. ANALYTIC SOLUTION

From (3.1) we obtain the equation

$$f(a;a+2) = f(a+1;a+2)f(a;a+1) \quad (4.1)$$

Combining this with the relation

$$f(a;a+1) = p(a+1) + q(a+1)f(a;a+2), \quad (4.2)$$

we obtain the recurrence relation

$$u(a) = p(a+1)/(1-q(a+1)u(a+1)), \quad (4.3)$$

where $u(a) = f(a; a+1)$, valid for $a = 1, 3, 5, \dots$.

It is easily seen that $u(b-2) = p(b-1)$. Thus the sequence $\{u(a)\}$ is determined.

Returning to (3.1), we see that

$$f(a; x) = \sum_{k=a}^{x-1} u(k). \quad (4.4)$$

5. INHOMOGENEOUS RANDOM WALK; SEMI-INFINITE INTERVAL

As our second example, consider the process described above in the case where $i = \infty$. We now wish to determine the expected value of the time spent by a particle starting at x at time zero in reaching the point a .

Let $f(a; x)$ denote the expected time. Then, as in § 3, we obtain the functional relation

$$f(a; x) = f(a+1; x) + f(a; a+1) \quad (5.1)$$

To obtain an analytic expression for $f(a; a+1)$, we combine the two expressions

$$\begin{aligned} f(a; a+1) &= p(a+1) + q(a+1)[f(a; a+2) + 1] \\ f(a; a+2) &= f(a+1; a+2) + f(a; a+1), \end{aligned} \quad (5.2)$$

and derive the relation

$$f(a; a+1) = \frac{1}{p(a+1)} + \frac{q(a+1)}{p(a+1)} f(a+1; a+2). \quad (5.3)$$

Iteration of this relation yields the infinite series

$$f(a; a+1) = \frac{1}{p(a+1)} + \frac{q(a+1)}{p(a+2)p(a+1)} + \frac{q(a+1)q(a+2)}{p(a+3)p(a+1)p(a+2)} + \dots \quad (5.4)$$

In order for this expression to converge, there must be a "drift" to the left. The condition $p(k) - q(k) \geq d > 0$ is clearly sufficient for the convergence of (5.4), but clearly much weaker conditions will suffice.

6. CHARACTERISTIC FUNCTIONS

Following the operational principle that characteristic functions can be determined by the same techniques which furnish expected values, let us introduce the function

$$g(a; x) = E(e^{1sz}), \quad (6.1)$$

where $z = z(a; x)$ is the random variable equal to the time spent by the particle in going from x to a .

Since $z(a; x) = z(a+1; x) + z(a; a+1)$, we have the functional equation

$$g(a; x) = g(a+1; x)g(a; a+1). \quad (6.2)$$

Furthermore, as above, the two equations

$$\begin{aligned} g(a; a+1) &= p(a+1)e^{1s} + q(a+1)e^{1s}g(a; a+2) \\ g(a; a+2) &= g(a+1; a+2)g(a; a+1) \end{aligned} \quad (6.3)$$

yield the recurrence relation

$$g(a;a+1) = p(a+1)e^{1s} / [1 - q(a+1)e^{1s} g(a+1;a+2)]. \quad (1.4)$$

7. DISCUSSION

The foregoing techniques can be extended to treat the case where there is a general distribution of jumps at each stage, and the case where we introduce time dependence by **requiring** the probability that the particle emerge at a on or before a given time T . These matters, as well as those mentioned in § 1, will be discussed in subsequent papers.

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